SOLUTION OF THE NONLINEAR EQUATION OF HEAT CONDUCTION BY INTEGRAL ESTIMATES

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First, the limited engineering applicability of the Biot method and of the heat-balance integral is demonstrated. A more general approach (integroenergetical method) is then shown, with integral estimates based on conservation laws.

The analysis of nonlinear transient heat conduction has been of concern to the experts in this field for a long time already. Along with numerical and seminumerical (zonal) methods, of undiminished interest are also approximate analytical solutions (1) which would completely eliminate unwieldy "direct counting" by computer or would limit the computations to auxiliary ones only.

Most analytical methods impose considerable restrictions on the nonlinearity, on the input signal, etc. For instance, in some cases it is possible to introduce a self-adjoint variable [2-4]. *

Among the more general methods, the Biot variational method [5-7] and the integral method [8-11] have in recent years been widely used for solving linear as well as nonlinear problems of transient heat conduction.

Integral Method. The many variants of this method reduce to an averaging of the field function with respect to the α-th moment by means of the integral transformation

$$\int_0^\infty \frac{\partial}{\partial x}\left[\lambda(\theta)\frac{\partial\theta}{\partial x}\right]x^\alpha\,dx = \frac{d}{dt}\int_0^\infty C\theta x^\alpha\,dx \tag{1}$$

(in thermotechnical notation).

With $\alpha = 0$ we have the well known heat-balance integral [9, 10].

It is quite evident that expression (1) accounts for the effects of nonlinearity only at the two extreme points: $\theta(0, t) = \theta_s$, $\theta(\infty, t) = 0$ and, when the field of some incompletely determinate function is approximated, makes it possible to select one universal coordinate. Thus, the integral method yields one "soft" condition and, therefore, is applicable to the solution of problems involving processes close to already known ones so that the approximating function $\theta(x, t)$ can be successfully selected.

Biot Method. This method, in the opinion of many authors [5-7], is the most accurate one, inasmuch as the Biot variational principle yields the equation of heat conduction. We note that this fact merely indicates the generality of the "least action" principle, while telling nothing about the true accuracy of approximate solutions obtained by the Biot method.

Being general and "exact" in principle, the Biot variational method requires for the solution of specific problems that the Hamiltonian be expressed in explicit form and, therefore, both the generality and the accuracy diminish by the usually arbitrary value assigned to the parameter s of the function profile

* Here and from now on references are cited regardless of the particular physical process (heat conduction, diffusion, etc.) which the nonlinear parabolic equation describes.

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which approximate the temperature field:

$$\theta = \theta_s(t)\left(1 - \frac{x}{q}\right)^s. \tag{2}$$

It is common practice to assume s = 2 [5, 7] or to select the value of s either on the basis of test data or by comparison with already known solutions [6, 9]. Meanwhile, by considering the equation

$$\frac{\partial}{\partial x}\left[\lambda(\theta)\frac{\partial\theta}{\partial x}\right] = C\frac{\partial\theta}{\partial t} \tag{3}$$

and stipulating a nonlinear thermal conductivity $\lambda(\theta) = p\theta^{p-1}$ $(p \geq 1)$, one can easily ascertain (on the basis of self-adjoint solutions, for example) that at the limit $sp \sim 1$ we have $\lim_{p\to\infty} s = 0$ and $q = \sqrt{2t}$, i. e., $s = s(p)$.

The physical significance of this "mathematical" incongruity is that solutions according to the Biot method with a stipulated value of s do not satisfy the condition of heat balance, even when λ = const. Thus, when the surface temperature of a semiinfinitely large body rises linearly with time ($\theta_s = t$), we have [7], letting $\lambda = C = 1$,

$$q = 2.29\sqrt{t},\ \theta = t(1 - x/2.29\sqrt{t})^2.$$

From this we obtain directly $H_s = 0.88\sqrt{t}$ and $\dot{Q} = 1.145\sqrt{t}$, but $0.88 \neq 1.145$ (a 26% discrepancy).

We recall that, according to the Biot method [7], one first inserts the expression $H = (d/dt)\int_0^q$ into the dissipative function in the variational equation, then determines the partial derivatives of the appropriate invariants with respect to q and $\dot{q}$ as with respect to independent variables, and thus obtains an ordinary differential equation in q(t). Meanwhile, q and $\dot{q}$ are related also through the equation of heat balance and, therefore, partial differentiation, which is valid for the exact solution (where q and $\dot{q}$ are related uniquely), is permissible here only at that definite value of s which satisfies the first integral of Eq. (3).

Integroenergetical Method. Since a variation of thermal flux is defined not only in universal coordinates θ_s and q but also by the profile ($Q = C\theta_s q/(s+1)$ in (2), for example), hence the profile parameter s of the approximating function is used as the third universal coordinate.

In the most general case, s is determined by the form of the $\lambda(\theta)$ nonlinearity as well as by the form of the input signal θ_s and can, therefore, also be directly a function of time: $s = s(\lambda, \theta_s, t)$. Usually $s = s(\lambda, \theta_s)$ = const and, for example, $s = s(p, n)$ when $\lambda = p\theta^{p-1}$ and $\theta_s = t^n$ (see later in the text).

In order to obtain an approximate solution, it is thus necessary to have three conditions: one exact condition at the boundary and two fundamental conditions which will be defined here.

We introduce the "thermal Poynting vector" $\overline{P}_\theta$ equal to the thermal flux density through a closed surface per unit time (the boundary of a half-space may serve as such a surface); by definition

$$\overline{P}_\theta = \overline{H}\theta. \tag{4}$$

In order to calculate $\overline{P}_\theta$, we multiply Eq. (3) by θdx and then integrate over the interval (0, q) on which the field exists. Letting $c\rho = C$ = const, which does not affect the generality of the results, and noting that $\theta(q, t) = 0$, we have now

$$-\lambda(\theta_s)\theta_x(0, t)\theta(0, t) = \int_0^q \lambda(\theta)\theta_x^2 dx + \frac{C}{2}\cdot\frac{d}{dt}\int_0^q \theta^2 dx. \tag{5}$$

Here and from now on we will use the following notation:

$$\frac{\partial u}{\partial x} = u_x,\ \frac{\partial u}{\partial x}dx = \partial_x u,\ u(0, t) = u_s(t)$$

will not write the argument t in denoting the values of the function at the surface, and will omit the subscript θ.

Since

$$-\lambda\theta_x = H,\ d/dt(C\theta) = -H_x,\ \partial_x(H\theta) = \partial_x\theta H + \partial_x H\theta, \tag{6}$$

hence (4), (5), and (6) yield

$$P_s = P_{\text{diss}} + P_{\text{pot}}. \tag{7}$$

Thus, (5), (6), and (7) are different forms of the equation of power balance in a transient temperature field.

As the second condition we use the first integral of Eq. (3):

$$-\lambda_s \theta_{x_s} = C \frac{d}{dt} \int_0^q \theta dx. \tag{8}$$

Relations (5)-(8) together with the boundary condition do completely define the approximating function in the form (2), for example. However, this is not the only significance of the derived relations. Considering that Eq. (8) yields a relation between q and $\dot{q}$ (which the Biot variational equation does not) and inserting (8) into the left-hand side of (5), we obtain

$$C\theta_s \frac{d}{dt} \int_0^q \theta dx = \int_0^q \lambda(\theta) \theta_x^2 dx + \frac{C}{2} \frac{d}{dt} \int_0^q \theta^2 dx. \tag{9}$$

It can be demonstrated that (9) is equivalent to the Biot variational equation, but in the latter the dissipative function is expressed in terms of $\dot{q}$ and the input inergy as well as the potential energy of the field are expressed in terms of q, while in (9), on the other hand, the dissipative function is rendered meaningless by the unique relation between q and $\dot{q}$ in system (5)-(8). All this will be confirmed in subsequent illustrative examples.

Thus, conditions (5)-(8) for approximating the field by an indeterminate function with three "degrees of freedom" (θ_S, q, s) as, for example, function (2) yield an approximate solution to the nonlinear equation of heat conduction which satisfies two conservation laws and the "least action" principle.

It is noteworthy that no stipulation has been made in this analysis as to the form of the nonlinearities or of the boundary conditions, provided that the latter can be reduced to an arbitrary (including also a nonlinear) input signal at the surface.

Three Examples Illustrating the Application of the Integroenergetical Method. We will now consider processes of transient heat conduction in a half-space with an internal nonlinearity $\lambda = \lambda(\theta)$ and with boundary conditions of the first or of the second kind, in response to an arbitrary input signal: either a temperature $\theta_S(t)$ or a thermal flux density $H_S(t)$; we will also solve the simplest problem for a plate so as to illustrate the effect which an "automatic" determination of the profile parameter s has when the process is divided into two ranges.

1. Let us find the transient temperature field in a half-space whose thermal conductivity is a function of the temperature, when the variation of the surface temperature is arbitrarily aperiodic (boundary condition of the first kind).

We let $\lambda = p\theta^{p-1}$ $(1 \leq p < \infty)$, C = 1, $\theta_S(t)$ be an arbitrary nondecreasing function of time, and $\dot{\theta}_S(t) \geq 0$. Using the approximation (2), we obtain from (5)

$$P_s = \frac{sp\theta_s^{p+1}}{q}; \quad P_{\text{diss}} = \frac{s^2 p \theta_s^{p+1}}{s(p+1)-1} \cdot \frac{1}{q}; \quad P_{\text{pot}} = \frac{d}{dt} \cdot \frac{q\theta_s^2}{2(2s+1)}. \tag{10}$$

Inserting the power components into (5) yields the equation

$$\frac{2sp(sp-1)(2s+1)}{s(p+1)-1} \cdot \frac{\theta_s^{p+1}}{q} = \frac{d}{dt}(q\theta_s^2),$$

which, after multiplication by $2q\theta_S^2$, is immediately integrated with respect to the variable $q\theta_S^2$. Finally, we have

$$q(t) = \sqrt{A_1 \int \theta_s^{p+3} dt / \theta_s^4}, \tag{11}$$

$$A_1 = \frac{4sp(sp-1)(2s+1)}{s(p+1)-1}.$$

Inserting (2) into (8) yields

$$sp(s+1)\theta_s^p / q = \frac{d}{dt}(q\theta_s). \tag{12}$$

TABLE 1. Comparative Summary of Solutions to the Nonlinear Equation of Heat Conduction

Method of solution	$\theta_s(t)$	p	s	Coef. for q(t)	$Q(t)$	ΔQ, %	Published source
Integroenergetical method	t	1	5,6	5	$0{,}76t^{3/2}$	+1,3	
Integral method	t	1	3	2,82	$0{,}705t^{3/2}$	—6	[10]
Biot method	t	1	2	2,29	$0{,}765t^{3/2}$	+2	[7]
Exact solution	t	1	—	—	$4/3\sqrt{\pi}\ t^{3/2}$	0	*
Integroenergetical method	$\sqrt{t}$	1	3,65	4,13	$0{,}888t$	—0,22	
Integral method	$\sqrt{t}$	1	3	3,46	$0{,}865t$	—2,8	[10]
Biot method	$\sqrt{t}$	1	2	2,81	$0{,}94t$	+5,8	[7]
Exact solution	$\sqrt{t}$	1	—	—	$\sqrt{\pi}/2t$	0	*
Integroenergetical method	t	2	1	$\sqrt{2}$	$\sqrt{2}/2\,t^2$	0	Fig. 1b
Integral method	t	2		not applicable			
Biot method	t	2	2	2,13	$0{,}711\,t^2$	+1	*
Self-adjoint solution (accurate)	t	2	1	1	$2\sqrt{2}\,t^2$	0	[3]
Integroenergetical method	t	7	0,170	0,556	$0{,}475\,t^{4,5}$	—3	
Self-adjoint solution (accurate)	t	7	0,166	0,576	$0{,}495t^{4,5}$	+1	[4]†
Exact solution 1	t	7			not available		

*Calculated by this author.

†Error of the self-adjoint solution estimated on the basis of the largest discarded term.

From here follows that

$$q(t)=\sqrt{A_2\int \theta_s^{p+1}dt/\theta_s^2}\ ,\ A_2=2sp(s+1). \tag{13}$$

We recall that (11) and (13) both yield the same function q(t). Eliminating this function, we obtain the "connecting equation" for s:

$$\frac{2(sp-1)(2s+1)}{[s(p+1)-1](s+1)}=\theta_s^2\frac{\int \theta_s^{p+1}dt}{\int \theta_s^{p+3}dt}. \tag{14}$$

From (2) and (13) we obtain the thermal flux

$$Q(t)=\sqrt{2sp/s+1\int \theta_s^{p+1}dt}\ , \tag{15}$$

where s is calculated from (14).

For most widely encountered input signals $\theta_s(t)$, the right-hand side of Eq. (14) is independent of time and this corresponds to the assumption s = s(p) = const made before solving the problem. In any case, this condition is satisfied by a function of the form

$$\theta_s(t)=(c_1+c_2t)^n, \tag{16}$$

where c_1, c_2, and n are nonnegative constants.

If the right-side of Eq. (14) is a strong function of time, then Eqs. (10) and (12) become incompatible and the solution must be sought in the form (2) with s = s(t). Then Eq. (14) becomes an integral equation for s(t) whose method of solution requires a separate analysis.

For comparison with well known solutions, we show here the result when $\theta_s = t^n$ $(0 \le n < \infty)$.

The connecting equation (14) yields

$$a_0s^2+a_1s+a_2=0, \tag{14a}$$

where

$$a_0=3n(p^2-1)+3p-1;\ a_1=n(p^2-5p-4)+p-4;\ a_2=n(1-p)-1.$$

The root s(p, n) of (14a) is then inserted into (13), (2), and (15). Evidently, the same solution is obtained by insertion into (11).

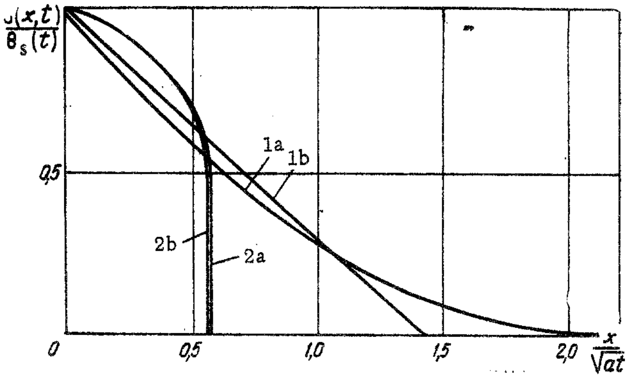

Fig. 1. Comparison between temperature profiles obtained for $\theta_s = 1$ and a nonlinear $\lambda(\theta)$ characteristic: 1a) by the Biot method with p = 2 and $\lambda = 2\theta$, 1b) by the integroenergetical method with p = 2 and $\lambda = 2\theta$, 2a) according to the self-adjoint solution with p = 7 and $\lambda = 7\theta^6$, 2b) by the integroenergetical method with p = 7 and $\lambda = 7\theta^6$.

2. Let us solve the same problem with a given thermal flux density at the surface: $H_s(t)$, $\dot{H}_s(t) > 0$ (boundary condition of the second kind). This problem reduces to the previous one, if θ_s is expressed in terms of H_s. Using (2), we obtain from (8)

$$-\theta_s = \frac{s+1}{q}\int H_s dt. \tag{17}$$

On the other hand, $-H(x, t) = \lambda(\theta)\theta_x(x, t)$ and from here

$$\theta_s = (qH_s/ps)^{1/p}. \tag{18}$$

Eliminating θ_s, we obtain

$$q(t) = \sqrt{A_{22}\left(\int H_s dt\right)^p / H_s}, \tag{19}$$

where

$$A_{22} = ps(s+1)^p.$$

Inserting (18) into (10) yields the equation

$$\frac{2(sp)^{1/p}(sp-1)(2s+1)}{s(p+1)-1} H_s^{\frac{p+1}{p}} = q^{-1/p}\frac{d}{dt}\left(q^{\frac{p+2}{2}} H_s^{\frac{2}{p}}\right), \tag{20}$$

which is multiplied by $(p + 1)/(p + 2)\,(q^{p+2/2}H_s^{2/p})^{-1/p+2}$ and then immediately integrated with respect to the variable in parentheses. Finally, we have

$$q = A_{12}H_s^{-2/p+2}\left(\int H_s^{\frac{p+3}{p+2}} dt\right)^{p/p+1}, \tag{21}$$

where

$$A_{12} = \left\{\frac{2(sp)^{1/p}(sp-1)(2s+1)(p+1)}{[s(p+1)-1](p+2)}\right\}^{p/p+1}.$$

From (19) and (21) follows the connecting equation for s:

$$\frac{2(sp-1)(2s+1)(p+1)}{[s(p+1)-1](p+2)} = H_s^{1/p+2}\frac{\int H_s dt}{\int H_s^{\frac{p+3}{p+2}} dt}. \tag{22}$$

3. Let us determine the field in an infinitely large plate whose thickness is b and $\lambda = C = 1$, when the temperature at the surface x = 0 changes stepwise from 0 to 1. For the first process range $\theta(x, 0) = 0$, $\theta(0, t) = 1$, $\theta(b, t) = 0$ we have from (14a);

$$2s^2 - 3s - 1 = 0,\ s_1 = 1.78,\ q = 3.16\sqrt{t},\ Q = 1.128\sqrt{t}.$$

For the second process range q = b = const, $\theta_b > 0$ we seek the temperature profile in the form

$$\theta(x, t) = \theta_b + (1 - \theta_b)\left(1 - \frac{x}{b}\right)^s. \tag{23}$$

Then

$$P_s = \frac{s(1-\theta_b)}{b};\quad P_{diss} = \frac{s^2(1-\theta_b)^2}{(2s-1)b};$$

$$P_{pot} = \frac{s\dot{\theta}_b(2s\theta_b+1)b}{(s+1)(2s+1)};\quad T = \frac{b^2}{s+1};\quad s \neq s_1.$$

Inserting the power components into (5), and (23) into (8), we obtain two equations:

$$\frac{1-\theta_b}{T} = \frac{2s-1}{2s+1}\cdot\frac{\dot{\theta}_b(2s\theta_b+1)}{s(1+\theta_b)-1},\quad \frac{1-\theta_b}{T} = \dot{\theta}_b,$$

which jointly yield the connecting equation:

$$\frac{2s-1}{2s+1} = \frac{s(1+\theta_b)-1}{2s\theta_b+1}. \tag{24}$$

From (24) we find $s_2 = 1.5$ and $T = b^2/2.5$ (the exact value is $T = 4b^2/\pi^2 = b^2/2.45$).

Discussion of the Results. Some solutions on the basis of the preceding examples, as special cases, are summarized in Table 1 and compared with already known solutions. For λ = const, moreover, the solutions are also compared with those according to Biot [5, 7] or Goodman [10]; for very nonlinear $\lambda(\theta)$ characteristics the comparison is made preferably with self-adjoint solutions [2, 3] as the most accurate ones (but providing almost no transition to the extreme case of linear problems). Concerning problems with a strong internal nonlinearity, we note that no solution by the Biot method could be found in the technical literature and that solutions by the Goodman method would obviously be unsuitable here.

Values of the function $\theta(x, t)$ calculated for two nonlinear problems are compared in Fig. 1.

In addition to the example given here, the method shown here was also applied to analogous problems for a plate, some cooling problems, and the calculation of transient processes in massive nonlinear magnets. It is interesting to note that in the last of these cases condition (5) becomes the low of momentum conservation:

$$i(0, t)B(0, t) = \int_0^q iB_x dx + \frac{1}{2\rho}\cdot\frac{d}{dt}\int_0^q B^2 dx, \tag{25}$$

where i(x, t) denotes the eddy current, B(x, t) denotes the magnetic induction, and ρ denotes the electrical resistivity of the ferromagnetic material.

In cases where the accuracy can be estimated directly, the error in determining the functions Q(t) and $\theta(x, t)$ did not exceed a few percent, except in the range of low $\theta(t, x)$ values at λ = const or in the case of a nonlinearity with a "constant" component ($\lambda = \lambda_0(1 + \alpha\theta)$, for example), a larger error being unavoidable there because of a finite q(t); this had almost no effect on the accuracy of the determination of the thermal flux Q(x, t).

The applicability of this method to problems involving heat sources and phase transformations, as well as the problem of designing an algorithm (where necessary) for it, would require further study.

NOTATION

θ	is the temperature;
$\lambda(\theta)$	is the thermal conductivity;
C	is the specific heat referred to volume;
H(x, t)	is the thermal flux density;
Q(t)	is the thermal flux;
q(t)	is the "zero front" (dimension of the thermal layer);
$\bar{P}_\theta$	is the "thermal Poynting vector," $W \cdot C/m^2$;
T	is the time constant;
p	is the nonlinearity parameter;

s	is the parameter (degree) of the approximating function;
x	is the length coordinate;
t	is the time;
b	is the plate thickness;
n	is the parameter (degree) of the parabola.

Subscripts

s	refers to the surface x = 0;
b	refers to the temperature at the back end surface (x = b);
A	refers to a constant which includes parameters p and s.

LITERATURE CITED

1. A. V. Lykov, Izv. Akad. Nauk SSSR, Otdel. Tekh. Nauk, Énergetika i Transport, No. 5 (1970).
2. Ya. B. Zel'sovich and Yu. P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena [in Russian], Fizmatgiz (1963).
3. G. I. Barenblatt, Prikl. Matem. i Mekhan., 16, No. 1 (1952).
4. I. D. Maergoiz, Izv. Akad. Nauk SSSR, Otdel. Tekh. Nauk, Énergetika i Transport, No. 5, 135 (1967).
5. T. I. Lardner, Rocket Engineering and Cosmonautics, No. 1, 225 (1963).
6. É. M. Gol'dfarb and O. S. Ereskovskii, Teplofiz. Vys. Temp., 4, No. 5 (1966).
7. G. F. Muchnik and I. B. Rubashov, Methods of Heat Transfer Theory [in Russian], Izd. Vysshaya Shkola (1970), Part 1.
8. G. I. Barenblatt, Prikl. Matem. i Mekhan., 18, No. 3 (1954).
9. A. I. Veinik, Approximate Calculation of Heat Conduction Processes [in Russian], Gosénergoizdat (1959).
10. T. R. Goodman, Trans. ASME Heat Transmission, 83, No. 1, 107 (1961).
11. I. D. Maergoiz, Izv. VUZov, Élektromekhanika, No. 6, 583 (1969).